23
Mathematics

As you live your life your attention is generally on the foreground things, and the background is usually taken for granted. We take for granted, most of the time, air, water, and many other things such as language and Mathematics. When you have worked in an organization for a long time its structure, its methods, its “ethos” if you wish, are usually taken for granted.

It is worth while, now and then, to examine these background things which have never held your close attention before, since great steps forward often arise from such actions, and seldom otherwise. It is for this reason we will examine Mathematics, though a similar examination of language would also prove fruitful. We have been using Mathematics without ever discussing what it is—most of you have never really thought about it, you just did the Mathematics—but Mathematics plays a central role in science and engineering.

Perhaps the favorite definition of Mathematics given by Mathematicians is:

> “Mathematics is what is done by Mathematicians, and Mathematicians are those who do Mathematics”.

Coming from a Mathematician its circularity is a source of humor, but it is also a clear admission they do not think Mathematics can be defined adequately. There is a famous book, *What is Mathematics,* and in it the authors exhibit Mathematics but do not attempt to define it.

Once at a cocktail party a Bell Telephone Laboratories Mathematics department head said three times to a young lady,

> Mathematics is nothing but clear thinking.

I doubt she agreed, but she finally changed the subject; it made an impression on me. You might also say

> Mathematics is the language of clear thinking.

This is not to say Mathematics is perfect—not at all—but nothing better seems to be available. You have only to look at the legal system and the income tax people, and their use of the natural language to express what they mean, to see how inadequate the English language is for clear thinking. This simple statement, “I am lying.” contradicts itself!

There are many natural languages on the face of the earth, but there is *essentially* only one language of Mathematics. True, the Romans wrote VII, the Arabic notation is 7 (of course the 7 is in the Latin form and not the Arabic) and the binary notation is 111, but they are all the same idea behind the surface notation. A7 is a7 is a7, and in every notation it is a prime number. The number 7 is not to be confused with its representation.

Most people who have given the matter serious thought have agreed if we are ever in communication with a civilization around some distant sun, then they will have essentially the same Mathematics as we do. Remember the hypothesis is we are in communication with them, which seems to imply they have developed to the state where they have mastered the equivalent of Maxwell's equations. I should note some philosophers have doubted even their communication system, let alone any details of it, would resemble ours in any way at all. But people who have their heads in the clouds all the time can imagine anything at all and are very seldom close to correct (witness some of the speculation the surface of the moon would have meters of dust into which the space vehicle would sink and suffocate the people).

The words "essentially equivalent" are necessary because, for example, their Euclidean geometry may include *orientation* and thus for the aliens two triangles may be congruent or anticongruent, Figure 23.I. Similarly, Ptolemy in his *Almagest* on astronomy used the sin x where we would use $2\sin(x/2)$, but essentially the idea is the same.

Over the many years there has developed five main schools of what Mathematics is, and not one has proved to be satisfactory.

The oldest, and probably the one most Mathematicians adhere to when they do not think carefully about it, is the *Platonic school*. Plato (427–347 B.C.) claimed the idea of a *chair* was more real than any particular chair. Physical chairs are subject to wear, tear, decay, and being lost; the ideal chair is immutable, eternal, so he said. Hence, he claimed, the world of ideas is more real than the physical world. The theorems of Mathematics, and all other such results, belong in this world of ideas (so Plato claimed) along with the numbers such as 7, and they have no existence in the physical world. You never saw, heard, touched, tasted, or smelled the abstract number 7. Yes, you have seen 7 horses, 7 cows, 7 chairs, but not the number 7 itself —a pure 7 uncontaminated by any particular realization. In an image Plato used, we see reality only as the shadows it casts on a wall. The true reality is never visible, only the shadows of truth come to our senses. It is our minds which transcend this limitation and reach the ideas which are the true reality, according to Plato.

Thus Platonic Mathematicians will say they "discovered" a result, not they "created" it. I "discovered" error correcting codes, rather than "created" them, if I am a Platonist. The results were always there waiting to be discovered, they were always possible.

The trouble with Platonism is it fails to be very believable, and certainly cannot account for how Mathematics evolves, as distinct from expanding and elaborating; the basic ideas and definitions of Mathematics have gradually changed over the centuries, and this does not fit well with the idea of the immutable Platonic ideas. Euler's (1707–1793) idea of continuity is quite differet from the one you were taught. You can, of course, claim the changes arise from our "seeing the ideas more clearly" with the passage of time. But when one considers non-Euclidean geometry, which arose from tampering with only the parallel postulate, and then think of the many other potential geometries which must exist in this Platonic space-every possible Mathematical idea and all the possible logical consequences from them must all exist in Plato's realm of ideas for all eternity! They were all there when the Big Bang happened!

A second major school of Mathematicians is the *formalists*. To them Mathematics is a formal game of starting with some strings of abstract symbols, and making permitted formal transformations on the strings much as you do when doing algebra. For them all of Mathematics is a mechanical game *where no interpretation of the meaning of the symbols is permitted* lest you make an all too human error. This school has Hilbert as its main protagonist. This approach to Mathematics is popular with the Artificial Intelligence people since that is what machines do *par excellence!*

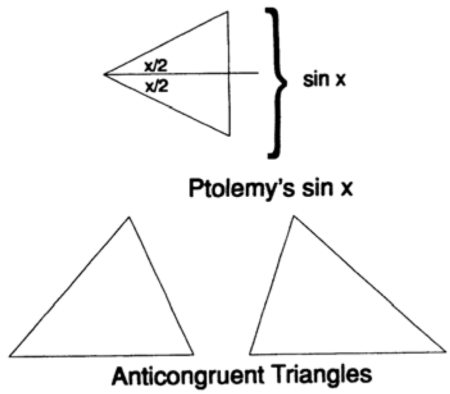

Figure 23.**I**

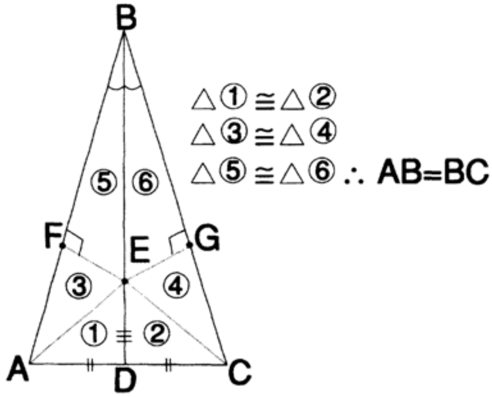

Figure 23.**II**

There was, probably, by the late Middle Ages (though I have never found just when it was first discovered) a well known proof, using classical Euclidean geometry, every triangle is isosceles. You start with a triangle ABC. Figure 23.II. You then bisect the angle at B and also make the perpendicular bisector of the opposite side at the point D. These two lines meet at the point E. Working around the point E you establish small triangles whose corresponding sides or angles are equal, and finally prove the two sides of the bisected angle are the same size! Obviously the proof of the theorem is wrong, but it follows the style used by classical Euclidean geometers so there is clearly something basically wrong. (Notice only by using metaMathematical reasoning did we decide Mathematical reasoning this time came to a wrong conclusion!)

To show where the false reasoning of this result arose (and also other possible false results) Hilbert examined, what Euclid had omitted to talk about, both *betweeness* and *intersections.* Thus Hilbert could show the indicated intersection of the two bisectors met outside the triangle, not inside as the drawing indicated. In doing this he added many more postulates than Euclid had originally given!

I was a graduate student in Mathematics when this fact came to my attention. I read up on it a bit, and then thought a great deal. There are, I am told, some 467 theorems in Euclid, but not one of these theorems turned out to be false after Hilbert's added his postulates! Yet, every theorem which needed one of these new postulates could not have been rigorously "proved" by Euclid! Every theorem which followed, and rested on such a theorem, was also not "proved" by Euclid. Yet the results in the improved system were still the same as those Euclid regarded as being true. How could this be? How could it be Euclid, though he had not actually proved the bulk of his theorems, never made a mistake? Luck? Hardly!

It soon became evident to me one of the reasons no theorem was false was that Hilbert "knew" the Euclidean theorems were "correct", and he had picked his added postulates so this would be true. But then I soon realized Euclid had been in the same position; Euclid knew the "truth" of the Pythagorean theorem, and many other theorems, and had to find a system of postulates which would let him get the results he knew in advance. Euclid did not lay down postulates and make deductions as it is commonly taught; he felt his way back from "known" results to the postulates he needed!

To paraphrase one of Hilbert's claims, *"When rigor enters, meaning departs."* The formalists claim there is no "meaning" in Mathematics—but if so why should society support Mathematics and Mathematicians? Why is it Mathematics has proved to be so useful? If there is no meaning in any place in all of Mathematics then why is it postulates and definitions are altered in time? The formalists simply cannot explain why Mathematics is in fact more than an idle game with no more meaning than the moves of chess.

Closely related to the formalists is the *logical school* who have tried to reduce all of Mathematics to a branch of logic. They, like every other school, have not been able to carry out their program— and for them it is more painful than for the others since they are supposed to be logicians! The famous Whitehead and Russell attempt, in three huge volumes, has generally been abandoned though large parts of their work has been retained. To use a famous quote from Russell:

> "Pure Mathematics consists entirely of assertions to the effect that, if such and such a proposition is true of *anything,* then such and such another proposition is true of that thing. It is essential not to discuss whether the first proposition is really true, and not to mention what the thing is, of which it is supposed to be true."

Here you see a blend of the logical and formalist schools, and the sterility of their views. The logicians failed to convince people their approach was other than an idle exercise in logic. Indeed, I will strongly suggest what is usually called the foundations of Mathematics is only the penthouse. A simple illustration of this is for years I have been saying if you come into my office and show me Cauchy's theorem is false, meaning it cannot derived from the usual assumptions, then I will certainly be interested, but in the long run I will tell you to go back and get new assumptions—I *know* Cauchy's theorem is "true". Thus, for me at least, Mathematics does not exclusively follow from the assumptions, but rather very often the assumptions follow from the theorems we "believe are true". I tend, as do many others, to group the formalists and logicians together.

Clearly, Mathematics is not the laying down of postulates and then making rigorous deduction from them the formalists pretend. Indeed, almost every graduate student in Mathematics has the experience they have to "patch up" the proofs of earlier great Mathematicians; and yet somehow the theorems do not change much, though obviously the great Mathematician had not really "proved" the theorem which was being patched up. It is true (though seldom mentioned) definitions in Mathematics tend to "slide" and alter a bit with the passage of time, so previous proofs no longer apply to the same statement of a theorem now we understand the words slightly differently.

The fourth school is the *intuitionists,* who boldly face this dilemma and ignore rigor. If you want absolute rigor, then, since we have had a rising standard of rigor, presumably no presently proved theorem is really "proved", rather the future will have to patch up our results, meaning we will not have "proved" anything! I suppose, if you want my position, I am partly an intuitionist. The above example about Cauchy's theorem illustrates my attitudeMathematics shall do what I want it to do. Contrary to Hermite (1822–1901) who said, “We are not the master but the servant of Mathematics”, I tend to believe (some of the time) we are the master. The postulates of Mathematics were not on the stone tablets Moses brought down from Mount Sinai; they are human made and hence subject to human changes as we please. Neither my view given above nor Hermite’s is exactly correct; the truth is a blend of them, we are both the master and the servant of Mathematics.

The nature of our language tends to force us into “yes-no”, something is or is not, you either have a proof or you do not. But once we admit there is a changing standard of rigor we have to accept some proofs are more convincing than other proofs. If you view proofs on a scale much like probability, running from 0 to 1, then all proofs lie in the range and very likely never reach the upper limit of 1, certainty.

The last major school is the *constructivists*. They insist you give explicit methods of constructing everything you talk about, and not proceed as the formalists do who say if a set of postulates is not proved to be inconsistent then the objects the postulates define “exist”. The constructivist’s approach can get you into a lot of trouble. There is no really rigorous basis for Mathematics for any of the other four schools, but the constructivists are too strict for many of our tastes since they exclude too much that we find valuable in practice. Computer scientists, excluding the AI people, tend to belong to the constructivist school, *if* they think about the matter at all.

Indeed, some numerical analysists tend to believe the “real number system” is the bit patterns in the computer—they are the true reality, so they say, and the Mathematician’s imagined number system is exactly that, “imagined”. Most users of Mathematics simply use it as a tool, and give little or no attention to their basic philosophy.

There is a group of people in software who believes we should “prove programs are correct” much as we prove theorems in Mathematics are correct. The two fallacies they commit are:

(1) we do not actually “prove” theorems!
(2) many important programming problems cannot be defined sharply enough so a proof can be given, rather the program which emerges defines the problem!

This does not mean there is nothing of value to their approach of proving programs are correct, only, as so often happens, their claims are much inflated.

Most Mathematicans belong to the Platonic school when they are doing Mathematics from day to day, but when pressed for a clear discussion of what they are doing they usually take refuge in the formalist school and claim Mathematics is an idle game with essentially no meaning to the symbols (not that they believe this, but it is a nice defensible position to adopt). They pretend they believe in the above quotation from Russell.

As you know from your courses in Mathematics, what you are actually doing, when viewed at the philosophical level, is almost never mentioned. The professors are too busy doing the details of Mathematics to ever discuss what they are actually doing—a typical technician’s behavior!

However, as you all know, Mathematics is remarkably useful in this world, and we have been using it without much thought. Hence we need more discussion on this background material you have used without benefit of thought.

The ancient Greeks believed Mathematics was “truth”. There was little or no doubt on this matter in their minds. What is more sure than 1+1=2? But recall when we discussed error correcting codes we said 1+1=0. This multiple use of the same symbols (you can claim the 1’s in the two statements are not the same things if you wish) contradicts logical usage. It was probably when the first non Euclidean geometries arose

Mathematicians came face to face with this matter that there could be different systems of Mathematics. They use the same words, it is true, such as points, lines, and planes, but apparently the *meanings* to be attached to the words differ. This is not new to you; when you came to the topic of *forces* in mechanics and to the addition of forces then you had to recognize scalar addition was not appropriate for vector addition. And the word “work” in physics is not the same as we generally mean in real life.

It would appear the Mathematics you choose to use must come from the field of application; Mathematics is not universal and “true”. How, then, are we to pick the right Mathematics for various applications? What meanings do the symbols of Mathematics have in themselves? Careful analysis suggests the “meaning” of a symbol only arises from how it is used and not from the definitions as Euclid, and you, thought when he defined points, lines and planes. We now realize his definitions are both circular and do not uniquely define anything; the meaning must come from the relationships between the symbols. It is just as in the interpretive language I sketched out in Chapter 4, the meaning of the instruction was contained in the subroutine it called—how the symbols were processed—and not in the name itself! In themselves the marks are just strings of bits in the machine and can have no meaning except by how they are used.

The Mathematician Dodson (Lewis Carroll), who wrote *Alice in Wonderland* and *Through the Looking Glass,* specialized in logic, and these two books are extensive displays of how meaning resides in the use. Thus Humpty Dumpty asserted when he used a word it meant what he wanted it to mean, neither more nor less; Alice felt words had meanings independent of their use, and should not be used arbitrarily.

By now it should become clear the symbols mean what we choose them to mean. You are all familiar with different natural languages where different words (labels) are apparently assigned to the same idea. Coming back to Plato; what is a chair? Is it always the same idea, or does it depend on context? At a picnic a rock can be a chair, but you do not *expect* the use of a rock in someone’s living room as a chair. You also realize any dictionary must be circular; the first word you look up must be defined in terms of other words—there can be no first definition which does not use words.

You may, therefore, wonder how a child learns a language. It is one thing to learn a second language once you know a first language, but to learn the first language is another matter—there is no first place to appeal for meaning. You can do a bit with gestures for nouns and verbs, but apparently many words are not so indicatable. When I point to a horse and say the word “horse”, am I indicating the name of the particular horse, the general name of horses, of quadrupeds, of mammals, of living things, or the color of the horse? How is the other person to know which meaning is meant in a particular situation? Indeed, how does a child learn to distinguish between the specific, concrete horse, and the more abstract class of horses?

Apparently, as I said above, meaning arises from the use made of the word, and is not otherwise defined. Some years back a famous dictionary came out and admitted they could not prescribe usage, they could only say how words were used; they had to be “descriptive” and not “prescriptive”. That there is apparently no absolute, proper meaning for every word made many people quite angry. For example, both the New Yorker book reviewer and the fictional detective Nero Wolfe were very irate over the dictionary.

We now see all this “truth” which is supposed to reside in Mathematics is a mirage. It is all arbitrary, human conventions.

But we then face the *unreasonable effectiveness of Mathematics*. Having claimed there was neither “truth” nor “meaning” in the Mathematical symbols, I am now stuck with explaining the simple fact Mathematics is used and is an increasingly central part of our society, especially in science and engineering. We have passed from absolute certain truth in Mathematics to the state where we see there is no meaning at all in the symbols—but we still use them! We put the meaning into the symbols as we convert the assumptions of the problem into Mathematical symbols, and again when we interpret the results. Hence we can use the same formula in many different situations—Mathematics is sort of a universal mental tool for clear thinking.

A fundamental paradox of life, well stated by Einstein, is *it appears the world is logically constructed*. This is the most amazing thing there is—the world can be understood logically and Mathematically. I would warn you, however, recent developments in basic physics casts some doubt on his remark, and this is discussed in the next chapter.

Supposing for the moment the above remark of Einstein is true, then the problem of applying Mathematics is simply to recognize an analogy between the formal Mathematical structure and the corresponding part of "reality". For example, for the error correcting codes I had to see for symbols of the code, if I were to use 0 and 1 for the basic symbols, and use a 1 for the position of an error (the error was simply a string of 0's with one 1 where the error occurred), then I could "add" the strings if and only if I chose 1+1=0 as my basic arithmetic. Two successive errors in the same position is the same as no error. I had to see an analogy between parts of the problem and a Mathematical structure which at the start I barely understood.

Thus part of the effectiveness of Mathematics arises from the recognition of the analogy, and only in so far as the analogy is extensive and accurate can we use Mathematics to predict what will happen in the real world from the manipulation of the symbols at our desks.

You have been taught a large number of these identifications between Mathematical models and pieces of reality. But I doubt these will cover all future developments. Rather, as we want, more and more, to do new things which are now possible due to technical advancements of one kind or another, *including understanding ourselves better,* we will need many other Mathematical models.

I suggest, with absolutely no proof, in the past we have found the easy applications of Mathematics, the situations where there is a close correspondence between the Mathematical structure and the part being modeled, and in the future you will have to be satisfied with poorer analogies between the two parts. We will, in time, I believe, want Mathematical models in which the whole is not the sum of the parts, but the whole may be much more due to the "synergism" between the parts. You are all familiar with the fact the organization you are in is often more than the total of the individuals—there is morale, means of control, habits, customs, past history, etc. which are indefinably separate from the particular individuals in the organization. But if Mathematics is clear thinking, as I said at the start of this chapter, then Mathematics will have to come to the rescue for these kinds of problems in the future. Or to put it differently, whatever clear thinking you do, especially if you use symbols, then that is Mathematics!

I want to close with even more disturbing thoughts. It is not evident, though many people, from the early Greeks on, implicitly act as if it were true, that all things, whatsoever they may be, can be put into words—you could talk about anything, the gods, truth, beauty, and justice. But if you consider what happens in a music concert, then it is obvious what is transmitted to the audience cannot be put into words—if it could then the composer and musicians would probably have used words. All the music critics to the contrary, what music communicates cannot (apparently) be put into words. Similarly, but to a lesser extent, for painting. Poetry is a curious field where words are used, but the true content of the poem is not in the words!

Similarly, the three things of Classic Greece, *truth, beauty and justice,* though you all think you know what they mean, cannot (apparently) be put into words. From the time of Hammurabi (1955–1913 B.C.) the attempt to put justice into words has produced the *law,* and often the law is not your conception of justice. There is the famous question in the Bible, "What is truth?" And who but a beauty judge would dare to judge "beauty"?

Thus I have gone beyond the limitations of Godel's theorem, which loosely states if you have a reasonably rich system of discrete symbols (the theorem does not refer to Mathematics in spite of the way it is usually presented) then there will be statements whose truth or falsity cannot be proved within the system. It follows if you add new assumptions to settle these theorems, there will be new theorems which you cannot settle within the new enlarged system. *This indicates a clear limitation on what discrete symbol systems can do.*

Language at first glance is just a discrete symbol system. When you look more closely, Godel's theorem supposed a set of definite symbols with unchanging meaning (though some may be context sensitive), but as you all know words have multiple meanings, and degrees of meaning. For example the word "tall" in a tall building, a tall person, or a tall tale, has not exactly the same meaning each time it occurs. Indeed, a tone of voice, a lift of an eyebrow, the wink of an eye, or even a smile, can change the meaning of what is being said. Thus language as we actually use it does not fit into the hypotheses of Godel's theorem, and indeed it just might be the reason language has such peculiar features is in life it is necessary to escape the limitations of Godel's theorem. We know so little about the evolution of language and the forces which selected one version over another in the survival of the fittest language, that we simply cannot do more than guess at this stage of knowledge of languages and the circumstances in which language developed and evolved.

The standard computers can presently handle discrete symbols (though what some neural networks handle may be another matter), and hence, apparently, there may be many things they cannot handle. As noted in Chapter 19, if you assume neural nets have a finite usable bandwidth then the sampling theorem gives you the equivalence of bandwidth and sampling rate.

I think in the past we have done the easy problems, and in the future we will more and more face problems which are left over and require new ways of thinking and new approaches. The problems will not go away—hence you will be expected to cope with them—and I am suggesting at times *you may have to invent new Mathematics* to handle them. Your future should be exciting for you if you will respond to the challenges in correspondingly new ways. Obviously there is more for the future to discover than we have discovered in all the past!